B.sc(H) part 2 paper 3
Topic:Homomophism &Isomorphism
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Homomorphism&Isomorphism

Defination: Let G be a group with respect to a binary operation o and let G' be another group with respect to a binary operation o'. Let $f : G \to G'$ be a mapping such that

$$f(a \; o \; b) = f(a) \; o' \; f(b)$$

where, $a, b \in G$ and $f(a)$ and $f(b)$ are their images under f. Then the mapping f is said to be an *homomorphism* and we say that G is homomorhic to G'.

If the mapping f is a one-one and onto mapping, then f is said to be an *isomorphism* and we say that G is isomorphic to G'.

Thus if f is an isomorphism, the following conditions are satisfied.

(i) f is a homomorphism, that is $f(a \; o \; b) = f(a) \; o' \; f(b)$

i.e. f preserves group operation.

(ii) f is a one-one and onto mapping.

Ex 1.. Let I be the additive group of integers and let E be the subgroup of even integers.

That is $G = (I, +)$ and $G' = (E, \cdot)$.

Consider the mapping $f : I \to E$ given by

$$f(n) = 2n \text{ where } n \in I.$$

Show that f is an isomorphism.

Soln. f preserves operations in G and G′.

Let $m, n \in I$. Then

$$f(m + n) = 2(m + n) = 2m + 2n$$

$$= f(m) + f(n)$$

f is onto : Also, f is an onto mapping, since an even integer say $2n \in E$ is the image of an integer $n \in I$.

f is one one : Again, f is a one-one mapping, for

$$f(m) = f(n) \Rightarrow 2m = 2n$$

i.e. ,, ,, $\Rightarrow m = n.$

Thus we find that (i) f is a homomorphism

and (ii) f is one-one and onto mapping.

Hence f is an isomorphism.

Ex. 2. Let Z be the additive group of integers and let G' be the multiplicative group of numbers of the form 2^m, where $m = 0, \pm 1, \pm 2, \ldots$

That is, $$G = (Z, +)$$

and $$G' = [\{2^m, m = 0, \pm 1, \pm 2, \ldots\},]$$

Let the mapping : $f : Z \rightarrow \{2^m\}$ be defined by

$$f(m) = 2^m; \quad m \in I.$$

Show that f is an isomorphism.

Soln. f preserves operation in G and G'.

Let $m, n \in I$. Then

$$f(m + n) = 2^{m+n} = 2^m \cdot 2^n$$

$$= f(m) \cdot f(n)$$

Therefore f is a homomorphism.

f is onto : Obviously f is an onto mapping, since the reimage-point of any element say $2^k \in G'$ is k which $\in I$.

f is one-one : Also f is one-one, since $f(m) = f(n) \Rightarrow 2^m = 2^n$, i.e. $m = n$

Hence f is an isomorphism.

Example of a homomorphism which is not isomorphism

Ex.1. Let $(Z, +)$ be the additive group of integers. Let m be a fixed integer. Show that the map $f : Z \to Z$ given by $f(a) = ma$, $a \in Z$ is a homomorphism.

Soln. Let $a, b \in Z$ Then

$$f(a + b) = m(a + b) = ma + mb = f(a) + f(b).$$

Hence f is a homomorphism.

But this homomorphism is one-one but not onto if $m \neq \pm 1$.

Ex.2 Let $(R, +)$ be the additive group of real numbers and $K = \{e^{i\theta}, \theta$ is real$\}$ be the multiplicative group of complex numbers with absolute value 1. Show that the map $f : R \to K$ given by $f(\theta) = e^{i\theta}$, $\theta \in R$ is a homomorphism.

Soln. Let $\theta_1, \theta_2 \in R$. Then

$$f(\theta_1 + \theta_2) = e^{i(\theta_1 + \theta_2)} = e^{i\theta_1} \cdot e^{i\theta_2} = f(\theta_1) \cdot f(\theta_2)$$

Hence f is a homomorphism.

But this homomorphism is onto but not one-one, because

$$f(\theta + 2n\pi) = e^{i(\theta+2n\pi)} = e^{i\theta} \cdot e^{i2n\pi}$$

$$= e^{i\theta} \cdot 1 \text{ for } n = 0, 1, 2, 3\ldots$$

In fact, if we take $\theta_1 = 2\pi$ and $\theta_2 = 4\pi$ then $\theta_1 \neq \theta_2$.

But $f(\theta_1) = e^{i2\pi} = 1$ and also $f(\theta_2) = e^{i4\pi} = 1$.

Thus $f(\theta_1) = f(\theta_2)$ although $\theta_1 \neq \theta_2$.

Hence f is not an isomorphism.